On diffusivity of a tagged particle in asymmetric zero-range dynamics

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Abstract

Consider a distinguished, or tagged particle in zero-range dynamics on Z^d with rate g whose finite-range jump probabilities p possess a drift $\sum_j jp(j) \neq 0$. We show, in equilibrium, that the variance of the tagged particle position at time t is at least order t in all $d \geq 1$ and at most order t in $d = 1$ and $d \geq 3$ for a wide class of rates g. Also, in $d = 1$, when the jump distribution p is totally asymmetric and nearest-neighbor, and also when the rate $g(k)$ increases and $g(k)/k$ decreases with k, we show the diffusively scaled centered tagged particle position converges to a Brownian motion.

Abbreviated title: On diffusivity of a tagged particle in zero-range.

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1 Introduction and Results

The zero-range process, introduced by Spitzer [20], follows the evolution of a collection of interacting random walks on $\mathbb{Z}^d$–namely, from a vertex with k particles, one of the particles displaces by j with rate $g(k)p(j)$. The function on the non-negative integers $g : \mathbb{N} \to \mathbb{R}_+$ is called the process "rate," and $p(\cdot)$ denotes the translation-invariant single particle transition probability. The above interaction is in the "time-domain" but not "spatially," hence the name "zero-range." We note the case when $g(k) \equiv k$ describes the situation of completely independent particles.

More precisely, let $\Sigma = \mathbb{N}^{\mathbb{Z}^d}$ be the configuration space where a configuration $\xi = \{\xi_i : i \in Z^d\}$ is given through occupation numbers ξ_i at vertex i. The zero-range system then is a Markov process $\xi(t)$ on the space of right-continuous paths with left limits $D(\mathbb{R}_+, \Sigma)$ with formal generator defined on real test functions ϕ,

$$(L\phi)(\xi) \ = \ \sum_j \sum_i g(\xi_i)p(j-i)(\phi(\xi^{i,j}) - \phi(\xi))$$

where $\xi^{i,j}$ is the configuration where a particle from i is moved to j. That is, $\xi^{i,j} = \xi - \delta_i + \delta_j$ where δ_k is the configuration with a single particle at k.

When a particle is distinguished, or tagged, we can consider the joint process $(x(t), \xi(t))$ on $D(\mathbb{R}_+, \mathbb{Z}^d \times \Sigma)$ where $x(t)$ is the position of the tagged particle at time t. The formal generator is given by

$$\begin{aligned}(\mathfrak{L}\psi)(x,\xi) &= \sum_j \sum_{i \neq x} g(\xi_i) p(j-i)(\psi(x, \xi^{i,j}) - \psi(x,\xi)) \\ &\quad + \sum_j g(\chi_x) \frac{\xi_x - 1}{\xi_x} p(j)(\psi(x, \xi^{x,x+j}) - \psi(x,\xi)) \\ &\quad + \sum_j \frac{g(\xi_x)}{\xi_x} p(j)(\psi(x+j, \xi^{x,x+j}) - \psi(x,\xi)).\end{aligned}$$

Here, the first term corresponds to particles other than at the tagged particle position x moving, the second term corresponds to other particles moving from x, and the last term represents motion of the tagged particle itself.

It will be convenient to consider the "reference" process from the point-of-view of the tagged particle, that is $\eta(t) = \{\xi_{i+x(t)}(t) : i \in \mathbb{Z}^d\}$ which can be obtained from the map $\pi((x(\cdot), \xi(\cdot))) = \eta(\cdot)$, and has formal generator

$$\begin{aligned}(\mathcal{L}\phi)(\eta) &= \sum_j \sum_{i \neq 0} g(\eta_i) p(j-i)(\phi(\eta^{i,j}) - \phi(\eta)) \\ &\quad + \sum_j g(\eta_0) \frac{\eta_0 - 1}{\eta_0} p(j)(\phi(\eta^{0,j}) - \phi(\eta)) \\ &\quad + \sum_j \frac{g(\eta_0)}{\eta_0} p(j)(\phi(\tau_j(\eta^{0,j})) - \phi(\eta)).\end{aligned}$$

Here, $\tau_j(\eta^{0,j})$ is the configuration obtained by displacing the tagged particle by j and then shifting the reference frame to its position; the notation $(\tau_j \eta)_k = \eta_{k+j}$ for $k \in \mathbb{Z}^d$.

The construction of these systems requires some conditions on g and p. Namely, we will assume throughout $g(0) = 0$, $g(k) > 0$ for $k \geq 1$, $|g(k+1) - g(k)| \leq K$ for some constant K, and $\liminf_{k \to \infty} g(k) > 0$, and also p is finite-range, that is $p(i) = 0$ for $|i| \geq R$ for some $1 \leq R < \infty$, whose symmetrization $s(x) = (p(x) + p(-x))/2$ is irreducible. Under weaker assumptions, which include the above, Andjel constructs the process $\xi(t)$ semigroup T_t^L on a class of "Lipschitz" functions $\mathcal{D}$ defined on a subset $\Sigma' \subset \Sigma$ of the configuration space,

$$\Sigma' = \left\{ \xi : \|\xi\| = \sum_{i \in \mathbb{Z}^d} |\xi_i| \beta_i < \infty \right\}$$

$$\mathcal{D} = \left\{ f : |f(\xi') - f(\xi'')| \leq c \|\xi' - \xi''\| \text{ for all } \xi', \xi'' \in \Sigma', \text{ for some } c = c(f) \right\}$$

where one can take $\beta_i = \sum_{n\geq 0} 2^{-n} s^{(n)}(i)$ for instance [1]. In a similar way, one can construct the process $(x(t), \xi(t))$ semigroup $T_t^{\mathfrak{L}}$ with respect to "Lipschitz" functions f where $|f(x,\xi') - f(y,\xi'')| \leq c[|x-y| + \|\xi' - \xi''\|]$ for all $x, y \in \mathbb{Z}^d$ and $\xi', \xi'' \in \Sigma'$. Then, also from the map π, process $\eta(t)$ semigroup $T_t^{\mathcal{L}}$ can be constructed on $\mathcal{D}$.

The zero-range process $\xi(t)$ has a well known explicit family product invariant measures $R_\alpha = \prod_{i\in Z^d} \mu_\alpha$ for $0 \leq \alpha < \liminf g(k)$ with marginal

$$\mu_\alpha(k) \;=\; \frac{1}{Z_\alpha}\frac{\alpha^k}{g(1)\cdots g(k)} \;\text{ when }\; k \geq 1 \quad\text{and}\; \mu_\alpha(0) \;=\; \frac{1}{Z_\alpha} \;\text{ when }\; k=0.$$

where Z_α is the normalization [1]. Let $\rho(\alpha) = \sum_k k\mu_\alpha(k)$ be the density of particles under R_α, and let $\rho^* = \lim_{\alpha\uparrow\liminf g(k)}$ [note that ρ^* may be finite for some type of g's]. As $\rho(\alpha) \uparrow \rho^*$ for $\alpha \uparrow \liminf g(k)$, for a given $0 \leq \rho < \rho^*$, there is a unique inverse $\alpha = \alpha(\rho)$.

For the reference process $\eta(t)$, the "palm" measures given by $dQ_\alpha = (\eta_0/\rho(\alpha))dR_\alpha$ are invariant (cf. [12], [15]). Only the marginal at the origin, which we denote μ_α^0, differs

$$\mu^0_\alpha(k) \;=\; \frac{1}{Z_\alpha}\frac{k}{\rho}\frac{\alpha^k}{g(1)\cdots g(k)} \;\text{ for }\; k \geq 1.$$

We now remark that, with respect to an invariant R_α, one can extend the zero-range process semigroup T_t^L and generator L to $L^2(R_\alpha)$ so that bounded functions in $\mathcal{D}$ form a core (cf. section 2 [16]). In the same way, with respect to a Q_α, the reference semigroup $T_t^{\mathcal{L}}$ and generator $\mathcal{L}$ can be extended to $L^2(Q_\alpha)$ with the same core. We note also here constructions of these processes can be made through the martingale-problem approach [15], [14]. Also, in this context, we note a Hille-Yosida type approach [8].

In addition, we note both families $\{R_\alpha\}$ and $\{Q_\alpha\}$ are in fact extremal measures in their respective convex set of invariant measures, and so process evolutions starting from such invariant states are time-ergodic [16]. Also, we note the adjoints L^* and $\mathcal{L}^*$ with respect to R_α and Q_α respectively correspond to "time-reversal" and are themselves zero-range and reference processes but with reversed jump probabilities $p^*(\cdot) = p(-\cdot)$. Finally, for $0 \leq \alpha < \liminf g(k)$, we note μ_α and μ_α^0 possess all moments.

In the following, to avoid degeneracies, we will work with a fixed $0 < \alpha < \liminf g(k)$ for which $\rho(\alpha) > 0$, and corresponding R_α and Q_α. For simplicity, we will denote by $E_\alpha[\cdot]$ and $P_\alpha[\cdot]$ the expectation and probability for the process measures starting from Q_α when there is no confusion; otherwise, the underlying measure will noted as a suffix.

We now discuss the problem studied in this article and its history. The question of tagged particle asymptotics was even mentioned in Spitzer's seminal paper. Such questions are important to physics and other applications [7]. What is known are some laws of large numbers (LLN), and some equilibrium central limit theorems (CLT) in "local balance" cases.

Write, with respect to the reference process, the position $x(t)$ as the sum total displacement "shift," that is

$$x(t) = \sum_j jN_j(t)$$

where $N_j(t)$ is the number of "shifts" of size j the reference process makes up to time t. The count $N_j(t)$ is compensated by $\int_0^t (g(\eta_0(s)/\eta_0(s))p(j)ds$, so that further

$$x(t) = \sum_j jM_j(t) + \sum_j j \int_0^t \frac{g(\eta_0(s))}{\eta_0(s)} p(j)ds \tag{1.1}$$

where $M_j(t) = N_j(t) - \int_0^t (g(\eta_0(s)/\eta_0(s))p(j)ds$ are orthogonal martingales. Moreover, we note $M_j^2(t) - \int_0^t (g(\eta_0(s))/\eta_0(s))p(j)ds$ are also martingales.

So, the tagged position $x(t)$ is a function of the reference process. For most of the paper, we will use this "reference frame" interpretation, that is, the notation $x(t)$ will denote the "compound shift" $\sum jN_j(t)$. It will also be useful to define

$$M(t) = \sum jp(j)M_j(t) \quad \text{and} \quad A(t) = \int_0^t \mathfrak{f}(\eta(s))ds$$

where $\mathfrak{f}(\eta) = (\sum jp(j))(g(\eta_0)/\eta_0) - (\alpha/\rho)$.

Then, in equilibrium, that is when the reference process is under initial distribution Q_α, one obtains

$$E_\alpha[x(t)] = E_\alpha[g(\eta_0)/\eta_0] \sum jp(j) = \frac{\alpha}{\rho(\alpha)} \sum jp(j)$$

and LLN

$$\lim_{t\to\infty} \frac{1}{t} x(t) = \frac{\alpha}{\rho(\alpha)} \sum_j jp(j) \quad \text{a.s.}$$

(cf. [15], [16]). Also, we refer the reader to some interesting LLN results under some non-equilibrium initial distributions [13].

With respect to fluctuations, when the jump probabilities are mean-zero, $\sum jp(j) = 0$, then $x(t) = \sum jM_j(t)$ is a martingale as the compensator terms cancel. Under equilibrium Q_α, the quadratic variation is

$$E_\alpha[|x(t)|^2] = \sum |j|^2 \int_0^t \frac{g(\eta_0(s))}{\eta_0(s)} p(j)ds \to \frac{\alpha}{\rho} \sum j^2 p(j) \quad \text{a.s.}$$

and so by martingale central limit theorem one gets the invariance principle

$$\lim_{\lambda\to\infty} \frac{1}{\sqrt{\lambda}} x(\lambda t) = B_\alpha(t)$$

where $B_\alpha(t)$ is d dimensional Brownian motion with covariance matrix $((\alpha/\rho)t \sum_j (e_i \cdot j)(e_k \cdot j)p(j))$ where $\{e_i\}$ is the standard basis of Z^d [15], [16].

The goal of this article is to further characterize the equilibrium fluctuations when the jump probability has a drift $\sum jp(j) \neq 0$. The first result is that the tagged particle variance is at least diffusive in all dimensions without conditions. As a comparison, we note this is not true for simple exclusion in the case $d = 1$ and the jump probability p is nearest-neighbor symmetric where the variance at time t is order $t^{1/2}$ [2].

Theorem 1 *Under initial distribution Q_α, we have in all dimensions $d \geq 1$ for $t \geq 0$ that*

$$\left[\frac{\alpha}{\rho(\alpha)} \sum_j |j|^2 p(j)\right] t \;\leq\; E_\alpha\left[|x(t) - E_\alpha[x(t)]|^2\right].$$

Proof. These bounds follow from an explicit calculation. We have

$$\begin{aligned}
E_\alpha[|x(t) - E_\alpha[x(t)]|^2] &= E_\alpha[|M(t) + A(t)|^2] \\
&= E_\alpha[|M(t)|^2] + 2E_\alpha[M(t)\cdot A(t)] + E_\alpha[|A(t)|^2] \qquad (1.2)\\
&= \frac{\alpha}{\rho}\sum |j|^2 p(j)t + 2\int_0^t E_\alpha[M(s)\cdot \mathfrak{f}(\eta(s))]ds + E_\alpha[|A(t)|^2].
\end{aligned}$$

Now, under time reversal at s, $\eta^*(u) = \eta(s-u)$, the number of j-shifts up to time s equals the number of $-j$-shifts in the reversed process up to time s, $N_j(s;\eta(\cdot)) = N_{-j}(s;\eta^*(\cdot))$. Also, $M^*(s) = \sum jN_{-j}(s) - \sum j\int_0^s (g(\eta_0^*(u))/\eta_0^*(u))p(j)ds$ is a martingale with respect to the reversed process. So, we have $E_\alpha[M(s)\cdot\mathfrak{f}(\eta(s))] = E_\alpha[M^*(s)\cdot \mathfrak{f}(\eta^*(0))] = 0$. Hence,

$$E_\alpha[|x(t) - E_\alpha[x(t)]|^2] \;=\; \frac{\alpha}{\rho}\sum |j|^2 p(j)t + E_\alpha[|A(t)|^2] \;\geq\; \frac{\alpha}{\rho}\sum |j|^2 p(j)t \qquad (1.3)$$

(we remark a different representation holds for exclusion processes [3]). □

To give some upperbounds on the tagged particle variance, we describe some classes of rate functions g.

Assumption (SP). Let L_n be the generator of the symmetric zero-range process on a cube $B_n = \{i \in \mathbb{Z}^d : |i| \leq n\}$, namely $(L_n\phi)(\xi) = \sum_{i,j\in B_n} g(\xi_i)(\phi(\eta^{i,j}) - \phi(\eta))s(j-i)$. Let $W(n,M)$ be the inverse of the spectral gap of L_n when there are M particles in B_n. Then, we assume the rate function g is such that there is a constant $C = C(\alpha,p,d)$ where $E_{R_\alpha}[(W(n,\sum_{i\in B_n}\xi_i))^2] \leq Cn^4$.

We observe rates g where $W(n,M) \leq Cn^2$ for a constant $C = C(d)$ independent of M, satisfy (SP) trivially, and include those rates where, for some $a \geq 1$ and $b > 0$, $g(k+a) - g(k) \geq b$ for all $k \geq 0$ [6]. Also, for the rate $g(k) = 1_{[k\geq 1]}$, it is known $W(n,M) \leq C(1+M/n)^2 n^2$ for some constant $C = C(d)$ [10], and so (SP) holds. It is most likely true that all rates g satisfy (SP).

Assumption (ID). The rate function g is such that $g(k)$ increases and $g(k)/k$ decreases with k.

Theorem 2 *Under initial distribution Q_α, when $\sum_i ip(i) \neq 0$, we have in $d = 1$ (without special assumptions), and in $d \geq 3$ under Assumption (SP) that there is a constant $C = C(\alpha, p, d)$ where for $t \geq 0$*

$$E_\alpha\bigg[|x(t) - E_\alpha[x(t)]|^2\bigg] \leq Ct.$$

We also note an invariance principle in a special case in $d = 1$.

Theorem 3 *Under initial distribution Q_α, in $d = 1$ when the jump probability is totally asymmetric $p(1) = 1$ and g satisfies Assumption (ID), we have the invariance principle*

$$\lim_{\lambda\to\infty} \frac{1}{\sqrt{\lambda}}\bigg(x(t) - E_\alpha[x(t)]\bigg) \ = \ B_\alpha(t)$$

where B_α is Brownian motion with diffusion coefficient $\sigma^2(\alpha) > \alpha/\rho$.

To compare, we remark with respect to asymmetric simple exclusion similar invariance principles have been shown in $d \geq 3$ for finite-range p [18] and in $d = 1$ when p is nearest-neighbor [5]. In this context, perhaps the main contribution of this paper is Theorem 2 in $d = 1$ as these upperbounds, valid for the general finite-range zero-range process, have no counterpart in the simple exclusion results.

The proof of Theorem 2 follows from an analysis of certain variance or H_{-1} norms, and is found in section 2. The proof of Theorem 3, in section 3, shows that a tagged particle has positively correlated increments in the totally asymmetric nearest-neighbor case in $d = 1$. Combined with diffusive variance bounds (Theorem 2), the invariance principle follows by applying a Newman-Wright theorem. We note, in comparison, a tagged particle in $d = 1$ simple exclusion with totally asymmetric nearest-neighbor transitions has negatively correlated increments, and in fact is a Poisson process [9]. In the zero-range context, since $\sigma^2(\rho) > \alpha/\rho = E_\alpha[x(1)]$, the tagged particle is not a Poisson process.

2 Proof of Theorem 2

We first discuss some definitions and estimates involving variational formulas for some resolvent quantities. Note that $\mathcal{D} \subset L^2(Q_\alpha)$ as for $f \in \mathcal{D}$ we have

$$\begin{aligned} E_\alpha[|f(\eta)|^2] \ &\leq \ c^2 E_\alpha\bigg[(\sum \eta_i\beta_i)^2\bigg] \\ &\leq \ c^2\sum_{i\neq j} E_\alpha[\eta_i\eta_j]\beta_i\beta_j + c^2\sum E_\alpha[\eta_i^2]\beta_i^2 \\ &\leq \ c^2 C\bigg(\sum \beta_i\bigg)^2 \ < \ \infty \end{aligned}$$

for a constant $C = C(\alpha)$. Also, we note the notation $E_\alpha[fg] = \langle f, g\rangle_\alpha$, and $E_\alpha[f^2] = \|f\|_0^2$.

The generator $\mathcal{L}$ can be decomposed into symmetric and anti-symmetric parts, $\mathcal{L} = \mathcal{S} + \mathcal{A}$ where $\mathcal{S} = (\mathcal{L}+\mathcal{L}^*)/2$ and $\mathcal{A} = (\mathcal{L}-\mathcal{L}^*)/2$. Consider the resolvent operator $(\lambda - \mathcal{L})^{-1} : L^2(Q_\alpha) \to L^2(Q_\alpha)$ well defined for $\lambda > 0$; in particular, $(\lambda - \mathcal{L})^{-1} f = \int_0^\infty e^{-\lambda s}(T_s^{\mathcal{L}} f) ds$. Since the symmetrization of $(\lambda - \mathcal{L})^{-1}$ has inverse $(\lambda - \mathcal{L}^*)(\lambda - \mathcal{S})^{-1}(\lambda - \mathcal{L}) = (\lambda - \mathcal{S}) + \mathcal{A}^*(\lambda - \mathcal{S})^{-1}\mathcal{A}$, we have the variational formula for $f \in \mathcal{D}$,

$$\langle f, (\lambda - \mathcal{L})^{-1} f\rangle_\rho \;=\; \sup_{\phi\in\mathcal{D}} \Big\{ 2\langle f, \phi\rangle_\alpha - \langle \phi, (\lambda - \mathcal{S})\phi\rangle_\alpha - \langle \mathcal{A}\phi, (\lambda - \mathcal{L})^{-1}\mathcal{A}\phi\rangle_\alpha \Big\}.$$

Now, as $\mathcal{A}^* = -\mathcal{A}$ and $\mathcal{A}^*(\lambda - \mathcal{S})^{-1}\mathcal{A}$ is a non-positive operator, we have the easy bound that $\langle f, (\lambda - \mathcal{L})^{-1} f\rangle_\alpha$ is bounded by its "symmetrization,"

$$\begin{aligned} \langle f, (\lambda - \mathcal{L})^{-1} f\rangle_\alpha &\le \sup_{\phi\in\mathcal{D}} \Big\{ 2\langle f, \phi\rangle_\alpha - \langle \phi, (\lambda - \mathcal{S})\phi\rangle_\alpha \Big\} \\ &= \langle f, (\lambda - \mathcal{S})^{-1}\rangle_\alpha. \end{aligned} \tag{2.1}$$

It will be convenient to define, for $f \in \mathcal{D}$, the $H_1(Q_\alpha)$ (semi)-norm by $\|f\|_1^2 = \langle f, (-\mathcal{L})f\rangle_\alpha = \langle f, (-\mathcal{S})f\rangle_\alpha$. The H_1 space then is the completion with respect to this norm. Explicitly, for $\psi \in \mathcal{D}$,

$$\begin{aligned} \langle \psi, (-\mathcal{S})\psi\rangle_\alpha &= \frac{1}{2}\sum_j \sum_{i\neq 0} E_\alpha[g(\eta_i)(\psi(\eta^{i,i+j}) - \psi(\eta))^2] s(j) \\ &\quad + \frac{1}{2}\sum_j E_\alpha[g(\eta_0)\frac{\eta_0 - 1}{\eta_0}((\psi(\eta^{0,j}) - \psi(\eta))^2] s(j) \\ &\quad + \frac{1}{2}\sum_j E_\alpha[\frac{g(\eta_0)}{\eta_0}(\psi(\tau_j(\eta^{0,j})) - \psi(\eta))^2] s(j). \end{aligned}$$

Let H_{-1} be the dual of H_1, namely, the completion over $\mathcal{D}$ with respect to norms $\|\cdot\|_{-1}$ given in terms of variational formulas

$$\|f\|_{-1}^2 \;=\; \sup_{g\in\mathcal{D}} \Big\{ 2\langle f, g\rangle_\alpha - \|g\|_1^2 \Big\} \;=\; \sup_{g\in\mathcal{D}} \frac{\langle f, g\rangle_\alpha}{\langle g, (-\mathcal{S})g\rangle_\alpha^{1/2}}. \tag{2.2}$$

Similarly, define the notation $\|f\|_{1,\lambda}^2 = \langle f, (\lambda - \mathcal{S})f\rangle_\alpha$ and $\|f\|_{-1,\lambda}^2 = \sup_{g\in\mathcal{D}}\{2\langle f, g\rangle_\alpha - \|g\|_{1,\lambda}^2\} = \langle f, (\lambda - \mathcal{S})^{-1} f\rangle_\alpha$. Note also, clearly as one can drop the $\lambda\langle f, f\rangle_\alpha$ term, that $\|f\|_{-1,\lambda}^2 \le \|f\|_{-1}^2$. In addition, we note the following useful "resolvent" estimate. For $f \in L^2(Q_\alpha)$ we have

$$\|f\|_{-1,\lambda}^2 \;=\; \int_0^\infty e^{-\lambda t}\langle T_t^{\mathcal{S}} f, f\rangle_\alpha dt \;\le\; \frac{1}{\lambda}\|f\|_0^2 \tag{2.3}$$

where $T_t^{\mathcal{S}}$ is the semigroup for the symmetrized process. Then, for $g \in \mathcal{D}$, we have

$$E_\alpha[fg] \;\le\; \|f\|_{-1,\lambda}\|g\|_{1,\lambda} \;\le\; \left[\frac{1}{\lambda}\|f\|_0^2\right]^{1/2}\|g\|_{1,\lambda}.$$

Now, for $f \in L^2(Q_\alpha)$, let $\sigma_t^2(f) = E_\alpha[(\int_0^t f(\eta(s))ds)^2]$, and observe from the decomposition (1.3), to get diffusive bounds on the tagged particle variance, one need only bound

$$\sigma_t^2(\mathfrak{h}) \;=\; E_\alpha\left[\left(\int_0^t \mathfrak{h}(\eta(s))ds\right)^2\right] \;<\; Ct$$

where $\mathfrak{h}(\eta) = (g(\eta_0)/\eta_0) - (\alpha/\rho)$ for some constant $C = C(\alpha, p, d)$. The next result relates $\sigma_t^2(f)$ to some H_{-1} norms.

Proposition 2.1 *For $f \in L^2(Q_\alpha)$, there is a universal constant C_1 such that*

$$\begin{aligned}\sigma_t^2(f) \;&\le\; C_1 t\langle f, (t^{-1} - \mathcal{L})^{-1} f\rangle_\alpha \\ &\le\; C_1 t\langle f, (t^{-1} - \mathcal{S})^{-1} f\rangle_\alpha \;\le\; C_1 t\|f\|_{-1}^2.\end{aligned}$$

Proof. The first line is well-known (with a proof found for instance in Lemma 3.9 [17]), the second bound is (2.1), and the third bound is explained after (2.2). □

Proof of Theorem 2. The strategy to bound $\sigma_t^2(\mathfrak{h})$ falls into two cases $d = 1$ and $d \ge 3$ under (SP). We first comment on the case $d = 1$, and then on the $d \ge 3$ case.

Case $d = 1$. (1) We will find a sequence of functions (in subsection 2.1.1) $\{\phi_\lambda : 0 < \lambda \le 1\} \subset \mathcal{D}$ such that

$$\sup_{0<\lambda\le 1} \|\mathfrak{h} - \mathcal{L}\phi_\lambda\|_{-1,\lambda} \;<\; \infty \tag{2.4}$$

and also

$$\sup_{0<\lambda\le 1}\left(\|\phi_\lambda\|_1^2 + \lambda\|\phi_\lambda\|_0^2\right) \;<\infty. \tag{2.5}$$

(2) Note $M_t(f) = f(\eta(t)) - f(\eta(0)) - \int_0^t(\mathcal{L}f)(\eta(s))ds$ is a martingale for $f \in \mathcal{D}$ with quadratic variation (by stationarity) $E_\alpha[(M_t(f))^2] = 2tE_\alpha[f(-\mathcal{L})f] = 2t\|f\|_1^2$. Then, we can write

$$-\int_0^t \mathcal{L}\phi_\lambda(\eta(s))ds \;=\; M_t(\phi_\lambda) + \phi_\lambda(\eta(0)) - \phi_\lambda(\eta(t))$$

and so (by stationarity)

$$\sigma_t^2(\phi_\lambda) \;\le\; 6\Big(t\|\phi_\lambda\|_1^2 + \|\phi_\lambda\|_0^2\Big) \;=\; 6t\Big(\|\phi_\lambda\|_1^2 + \frac{1}{t}\|\phi_\lambda\|_0^2\Big).$$

(3) Hence, by choosing $\lambda = t^{-1}$, we have from Proposition 2.1 that

$$
\begin{aligned}
\sigma_t^2(\mathfrak{h}) &\leq 2\sigma_t^2(\mathfrak{h} - \mathcal{L}\phi_{t^{-1}}) + 2\sigma_t^2(\mathcal{L}\phi_{t^{-1}}) \\
&\leq 2C_1 t\|\mathfrak{h} - \mathcal{L}\phi_{t^{-1}}\|_{-1,t^{-1}}^2 + 12t\bigg(\|\phi_{t^{-1}}\|_1^2 + \frac{1}{t}\|\phi_{t^{-1}}\|_0^2\bigg). \qquad (2.6)
\end{aligned}
$$

Then, by (1) and (2), $\sigma_t^2(\mathfrak{h}) \leq Ct$ for some constant $C = C(\alpha, p)$ and $t \geq 1$. For $0 \leq t < 1$, bounds are immediate. This finishes the proof in this case.

Case $d \geq 3$ and (SP). By Lemma 2.1, we need only show $\|\mathfrak{h}\|_{-1} < \infty$. One may be able to do this directly by "integration-by-parts" but as the Q_α marginal at the origin differs from the other marginals, one cannot apply immediately results in the literature. So, we "modify" the function $\mathfrak{h}$ and then apply these results.

Let j_0 be a point in the support of p. Consider the function $\phi(\eta) = (\eta_{j_0} - \rho)/(\rho p(j_0)) \in \mathcal{D}$. In subsection 2.1.2, we show that $\|\mathfrak{h} - \mathcal{L}\phi\|_{-1} < \infty$. Clearly $\|\phi\|_1 < \infty$ and $\|\phi\|_0 < \infty$. Then, by following the sequence (2.6), we have

$$
\sigma_t^2(\mathfrak{h}) \;\leq\; 2C_1 t\|\mathfrak{h} - \mathcal{L}\phi\|_{-1} + 12t\bigg(\|\phi\|_1^2 + \frac{1}{t}\|\phi\|_0^2\bigg) \;<\; Ct
$$

for a constant $C = C(\alpha, p, d)$ and $t \geq 1$. Bounds when $0 \leq t < 1$ are clear. This finishes the proof. $\square$

2.1 Some Estimates

We now turn to supplying the needed estimates in the two cases. We first make a calculation valid in any dimension $d \geq 1$. Let

$$
\phi(\eta) \;=\; \sum_{i \in Z^d} a_i(\eta_i - \rho)
$$

where $\sum a_i^2 < \infty$. Clearly $\phi \in L^2(Q_\alpha)$, and $\phi = \lim \phi^n$ is the $L^2(Q_\alpha)$ limit of functions $\phi^n = \sum_{|i| \leq n} a_i(\eta_i - \rho) \in \mathcal{D}$. Also, computes $\mathcal{L}\phi$ as the $L^2(Q_\alpha)$ limit $\mathcal{L}\phi = \lim \mathcal{L}\phi^n$ as bounded functions in $\mathcal{D}$ are a core. To this end, for n large, observe

$$
\phi^n(\eta^{i,i+j}) - \phi^n(\eta) \;=\; a_{i+j} - a_i
$$

and, as $\tau_j(\eta^{0,j}) = \tau_j(\eta + \delta_j - \delta_0) = \tau_j\eta + \delta_0 - \delta_{-j}$, and so $\phi^n(\tau_j\eta^{0,j}) = \sum_{|i| \leq n} a_i(\eta_{i+j} - \rho) + a_0 - a_{-j}$, we have

$$
\phi^n(\tau_1\eta^{0,1}) - \phi^n(\eta) \;=\; \sum_{|i| \leq n} (a_i - a_{i+j})(\eta_{i+j} - \rho) + (a_0 - a_{-j}).
$$

These computations enable us to write

$$
(\mathcal{L}\phi)(\eta) \;=\; \sum_j \sum_{i \neq 0} (a_{i+j} - a_i)g(\eta_i)p(j) + \sum_j (a_j - a_0)g(\eta_0)\frac{\eta_0 - 1}{\eta_0}p(j)
$$

$$
\begin{aligned}
& -\sum_j \sum_i (a_{i+j}-a_i)(\eta_{i+j}-\rho)\frac{g(\eta_0)}{\eta_0}p(j) + \sum_j (a_0-a_{-j})\frac{g(\eta_0)}{\eta_0}p(j) \\
= & \sum_j \sum_{i\neq 0,-j} (a_{i+j}-a_i)[g(\eta_i) - \eta_{i+j}\frac{g(\eta_0)}{\eta_0}]p(j) \\
& + \sum_j (a_0-a_{-j})[g(\eta_{-j}) - g(\eta_0)\frac{\eta_0-1}{\eta_0}]p(j) \\
& + \sum_j (a_j-a_0)g(\eta_0)[1-\frac{\eta_j+1}{\eta_0}]p(j).
\end{aligned}
$$

Here, we used $\sum_j \sum_i (a_{i+j}-a_i) = 0$ to reduce the first sum in the second line.

We now note the following basic useful computations.

Lemma 2.1 *Let $k \in \mathbb{Z}^d$ be a non-zero vertex, $k \neq 0$. Then, for $\psi \in L^2(Q_\alpha)$, we have*

$$
E_\alpha[g(\eta_k)\psi(\eta)] = \alpha E_\alpha[\psi(\eta+\delta_k)],\ \ E_\alpha[g(\eta_0)\frac{\eta_0-1}{\eta_0}\psi(\eta)] = \alpha E_\alpha[\psi(\eta+\delta_0)],
$$

and

$$
E_\alpha[\frac{g(\eta_0)}{\eta_0}\psi(\eta)] = E_\alpha[\frac{g(\eta_0)}{\eta_0}\psi(\tau_j(\eta^{0,j}))],\ E_\alpha[(\eta_j+1)\frac{g(\eta_0)}{\eta_0}\psi(\eta)] = E_\alpha[g(\eta_0)\psi(\tau_{-j}(\eta^{0,-j}))].
$$

Proof. We show the last equality as the others are similar. Write

$$
\begin{aligned}
E_\alpha[\psi(\eta)(\eta_j+1)\frac{g(\eta_0)}{\eta_0}] &= \frac{\alpha}{\rho}E_{P_\alpha}[\psi(\eta+\delta_0)(\eta_j+1)] = \frac{1}{\rho}E_{P_\alpha}[g(\eta_j)\psi(\eta^{j,0})\eta_j] \\
&= E_\alpha[g(\eta_0)\psi((\tau_{-j}\eta)^{j,0}] = E_\alpha[g(\eta_0)\psi(\tau_{-j}(\eta^{0,-j}))].
\end{aligned}
$$

□

Let now $\psi \in \mathcal{D}$ be a function. We can write, with Lemma 2.1,

$$
\begin{aligned}
E_\alpha[(\mathcal{L}\phi)\psi] &= \sum_j \sum_{i\neq 0,-j} (a_{i+j}-a_i)E_\alpha[(g(\eta_i)-\eta_{i+j}\frac{g(\eta_0)}{\eta_0})\psi(\eta)]p(j) \\
& \quad +\alpha\sum_j (a_0-a_{-j})E_\alpha[\psi(\eta+\delta_{-j})-\psi(\eta+\delta_0)]p(j) \\
& \quad +\sum_j (a_j-a_0)E_\alpha[g(\eta_0)(\psi(\eta)-\psi(\tau_{-j}(\eta^{0,-j})))]p(j). \qquad (2.7)
\end{aligned}
$$

It will be convenent, for later purposes, to observe that in the above computation we can take $E_\alpha[\psi] = 0$ without loss of generality as $E_\alpha[\mathcal{L}\phi] = 0$.

2.1.1 Estimates in $d = 1$. We now work in dimension $d = 1$, and choose the sequence

$$
a_i = \begin{cases} 0 & \text{for } i \leq 0 \\ c(1-\lambda)^{i-1} & \text{for } i \geq 1 \end{cases}
$$

with $c = \rho^{-1}$. For ease of notation, define $\nabla a_{k,j} = a_k - a_j$ and note

$$\nabla a_{i+j,i} = \begin{cases} 0 & \text{for } i, i+j \leq 0 \\ c(1-\lambda)^{i+j-1} & \text{for } i+j \geq 1 \text{ and } i \leq 0 \\ -c(1-\lambda)^{i-1} & \text{for } i \geq 1 \text{ and } i+j \leq 0 \\ c[(1-\lambda)^j - 1](1-\lambda)^{i-1} & \text{for } i, i+j \geq 1. \end{cases}$$

Clearly $|\nabla a_{i+j,i}| \leq |c|$ for all $i, j \in \mathbb{Z}$.

Recall now the range R of the distribution p, and write, with Lemma 2.1,

$$\begin{aligned} E_\alpha[(\mathcal{L}\phi)(\eta)\psi(\eta)] &= \sum_j \sum_{i \geq R+1} \nabla a_{i+j,i} E_\alpha[(g(\eta_i) - \eta_{i+j}\frac{g(\eta_0)}{\eta_0})\psi(\eta)]p(j) \\ &\quad + \sum_j \sum_{\substack{|i| \leq R \\ i \neq 0, -j}} \nabla a_{i+j,i} E_\alpha[(g(\eta_i) - \eta_{i+j}\frac{g(\eta_0)}{\eta_0})\psi(\eta)]p(j) \\ &\quad + \sum_j \nabla a_{0,-j} E_\alpha[g(\eta_0)\frac{\eta_0 - 1}{\eta_0}(\psi(\eta^{0,-j}) - \psi(\eta))]p(j) \\ &\quad + \sum_j \nabla a_{j,0} E_\alpha[g(\eta_0)(\psi(\eta) - \psi(\tau_{-j}\eta^{0,-j}))]p(j) \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Consider now the term I_1. Since $E_\alpha[\psi] = 0$, we can write

$$\begin{aligned} I_1 &= \sum_j \sum_{i \geq R+1} \nabla a_{i+j,i} E_\alpha[(g(\eta_i) - \alpha) - (\eta_{i+j} - \rho)\frac{g(\eta_0)}{\eta_0})\psi(\eta)]p(j) \\ &\quad -\rho \sum_j \sum_{i \geq R+1} \nabla a_{i+j,i} E_\alpha[\frac{g(\eta_0)}{\eta_0}\psi(\eta)]p(j) \\ &= J_1 + J_2. \end{aligned}$$

Note the last term J_2 equals, using $E_\alpha[\psi] = 0$ again,

$$\begin{aligned} J_2 &= \rho a_{R+1} E_\alpha[(g(\eta_0)/\eta_0)\psi(\eta)] \\ &= E_\alpha[(g(\eta_0)/\eta_0)\psi(\eta)] + ((1-\lambda)^R - 1)E_\alpha[(g(\eta_0)/\eta_0)\psi(\eta)] \\ &= E_\alpha[\mathfrak{h}\psi(\eta)] + J_3. \end{aligned}$$

Hence, we have that

$$E_\alpha[(\mathfrak{h} - \mathcal{L}\phi)\psi] = -(I_2 + I_3 + I_4 + J_1 + J_3). \tag{2.8}$$

To show the bound in (2.4), by the variational characterization of $\|\cdot\|_{-1,\lambda}$ (cf. (2.2)), we need only verify

$$|I_2 + I_3 + I_4 + J_1 + J_3| \leq C\|\psi\|_{1,\lambda}$$

for some constant $C = C(\alpha, p)$.

To this end, observe, by Schwarz inequality,

$$|I_3| \leq \Big(\sum_j |c|^2 p(j)\Big)^{1/2}\Big(\sum_j \alpha E_\alpha[g(\eta_0)\frac{\eta_0 - 1}{\eta_0}(\psi(\eta^{0,-j}) - \psi(\eta))^2]p(j)\Big)^{1/2} \leq C\|\psi\|_1$$

for a constant $C = C(\alpha)$ as $p(j) \leq 2s(-j)$. Also,

$$\begin{aligned} |I_4| &\leq \Big(\sum_j |c|^2 p(j)\Big)^{1/2}\Big(\sum_j E_\alpha[g(\eta_0)\eta_0]E_\alpha[\frac{g(\eta_0)}{\eta_0}(\psi(\eta) - \psi(\tau_{-j}(\eta^{0,-j})))^2]p(j)\Big)^{1/2} \\ &\leq 2|c|\sqrt{E_\alpha[g(\eta_0)\eta_0]}\|\psi\|_1^2. \end{aligned}$$

For the second term I_2, note, with Lemma 2.1,

$$\begin{aligned} &E_\alpha[(g(\eta_i) - \eta_{i+j}\frac{g(\eta_0)}{\eta_0})\psi(\eta)] \\ &\quad = E_\alpha[g(\eta_0)\frac{\eta_0 - 1}{\eta_0}(\psi(\eta^{0,i}) - \psi(\eta))) + (g(\eta_0) - g(\eta_0)\frac{\eta_{i+j} + 1}{\eta_0})\psi(\eta)] \\ &\quad = E_\alpha[g(\eta_0)\frac{\eta_0 - 1}{\eta_0}(\psi(\eta^{0,i}) - \psi(\eta))) + g(\eta_0)(\psi(\eta) - \psi(\tau_{-(i+j)}(\eta^{0,-(i+j)})))]. \end{aligned}$$

Then,

$$\begin{aligned} |I_2| \leq &\Big(\sum_j \sum_{\substack{|i|\leq R \\ i\neq 0,-j}} |c|^2 p(j)\Big)^{1/2}\Big(\sum_j \sum_{|i|\leq R} 2\alpha E_\alpha[g(\eta_0)\frac{\eta_0 - 1}{\eta_0}(\psi(\eta^{0,i}) - \psi(\eta)))^2]p(j) \\ &+2E_\alpha[g(\eta_0)\eta_0]E_\alpha[\frac{g(\eta_0)}{\eta_0}(\psi(\eta) - \psi(\tau_{-(i+j)}(\eta^{0,-(i+j)})))^2]p(j)\Big)^{1/2}. \end{aligned}$$

Note, as s is irreducible, $u \in \mathbb{Z}$ can be written $u = \sum_{k=1}^m l_k$ for points l_k in the support of s, $s(l_k) > 0$. Let $r_0 = 0$ and $r_k = \sum_{n=1}^k l_n$ for $1 \leq k \leq m$. Then, with Lemma 2.1,

$$\begin{aligned} |E_\alpha[g(\eta_0)\frac{\eta_0 - 1}{\eta_0}(\psi(\eta^{0,u}) - \psi(\eta)))^2]| &= \alpha|E_\alpha[(\psi(\eta + \delta_u) - \psi(\eta + \delta_0))^2]| \\ &\leq m\alpha\sum_{k=0}^{m-1} E_\alpha[(\psi(\eta + \delta_{r_k}) - \psi(\eta + \delta_{r_{k+1}}))^2] \\ &\leq C\|\psi\|_1^2 \end{aligned}$$

for some constant $C = C(p, m)$ as p is finite-range. Also,

$$|E_\alpha[\frac{g(\eta_0)}{\eta_0}(\psi(\eta) - \psi(\tau_u(\eta^{0,u})))^2]|$$

$$\le m \sum_{k=0}^{m-1} E_\alpha[\frac{g(\eta_0)}{\eta_0}(\psi(\tau_{r_k}(\eta^{0,r_k})) - \psi(\tau_{r_{k+1}}(\eta^{0,r_{k+1}})))^2]$$
$$= m \sum_{k=0}^{m-1} E_\alpha[\frac{g(\eta_0)}{\eta_0}(\psi(\eta) - \psi(\tau_{l_{k+1}}(\eta^{0,l_{k+1}})))^2] \ \le \ C\|\psi\|_1^2$$

for a $C = C(p, m)$ again as p is finite-range. Then, as the sums in I_2 are finite, I_2 is bounded $|I_2| \le C\|\psi\|_1^2$ for some constant $C = C(\alpha, p)$.

To bound J_1, we use the resolvent bound (2.3). Namely, as $\{g(\eta_i) - \alpha : i \in \mathbb{Z}^d\}$ is an orthogonal family,

$$\begin{aligned}
&|\sum_j \sum_{i \ge R+1} \nabla a_{i+j,i} E_\alpha[(g(\eta_i) - \alpha)\psi(\eta)]p(j)| \\
&\quad \le \|\sum_j \sum_{i \ge R+1} \nabla a_{i+j,i}(g(\eta_i) - \alpha)p(j)\|_{-1,\lambda}\|\psi\|_{1,\lambda} \\
&\quad \le \left(\frac{1}{\lambda}\sum_j \sum_{i \ge R+1} \nabla^2 a_{i+j,i} E_\alpha[(g(\eta_1) - \alpha)^2]p(j)\right)^{1/2} \|\psi\|_{1,\lambda}
\end{aligned}$$

and, as $\{(\eta_{i+j} - \rho)\frac{g(\eta_0)}{\eta_0} : i \ge R+1\}$ is also an orthogonal collection,

$$\begin{aligned}
&|\sum_j \sum_{i \ge R+1} \nabla a_{i+j,i} E_\alpha[(\eta_{i+j} - \rho)\frac{g(\eta_0)}{\eta_0})\psi(\eta)]p(j)| \\
&\quad \le \left(\frac{1}{\lambda}\sum_j \sum_{i \ge R+1} \nabla^2 a_{i+j,i} E_\alpha[(\eta_1 - \rho)^2 \frac{g^2(\eta_0)}{\eta_0^2}]p(j)\right)^{1/2} \|\psi\|_{1,\lambda};
\end{aligned}$$

then, $|J_1| \le C(\lambda^{-1}\sum_{i \ge 1} \nabla^2 a_{i+1,i})\|\psi\|_{1,\lambda}$ for some $C = C(\alpha, p)$ as $i + j \ge 1$ for $i \ge R+1$ beyond the range of p.

Finally, J_3 is bounded by the resolvent bound (2.3)

$$|J_3| \ \le \ \left[\frac{|(1-\lambda)^R - 1|}{\lambda}\|g(\eta_0)/\eta_0\|_0^2\right]^{1/2} \|\psi\|_{1,\lambda}.$$

Putting these estimates together, using a form of Schwarz–relation $2ab = \inf_\epsilon \epsilon^{-1}a^2 + \epsilon b^2$–we obtain, for a constant $C = C(\alpha, p)$,

$$|I_2 + I_3 + I_4 + J_1 + J_3| \ \le \ C\left(1 + \frac{1}{\lambda}\sum_{i \ge 1} \nabla^2 a_{i+1,i}\right)^{1/2} \|\psi\|_{1,\lambda}.$$

Now, by direct computation, we have that

$$\sum_{i \ge 1} a_i^2 \ = \ \frac{|c|^2}{\lambda} \quad \text{and} \quad \frac{1}{\lambda}\sum_{i \ge 1} \nabla^2 a_{i+1,i} \ = \ \frac{\lambda^2 |c|^2}{\lambda}\frac{1}{\lambda(2-\lambda)}$$

which shows (2.4) via (2.8).

To show (2.5), we observe $\lambda\|\phi\|_{L^2}^2 = \lambda(|c|^2/\lambda) = |c|^2$ and

$$\begin{aligned}\|\phi\|_1^2 &= \alpha\sum_j\sum_i \nabla^2 a_{i+j,i}s(j)\\ &\quad+\sum_j E_\alpha[\frac{g(\eta_0)}{\eta_0}\Big(\sum_i \nabla a_{i,i+j}(\eta_{i+j}-\rho)+\sum_j \nabla a_{0,-j}\Big)^2]p(j)\\ &\le C + C\sum_j\sum_{i\ge 1}\nabla^2 a_{i+j,i} \ \le\ C'\end{aligned}$$

for some constants $C = C(\alpha, p)$ and $C' = C'(\alpha, p)$ using, as before, the orthogonality of $\{(g(\eta_0)/\eta_0)(\eta_{i+j} - \rho)\}$.

2.1.2 Estimates in $d \geq 3$ under (SP). From (2.7), for the function $\phi(\eta) = (\eta_{j_0} - \rho)/(\rho p(j_0))$ and $\psi \in \mathcal{D}$, we have $\rho p(j_0)E_\alpha[(\mathcal{L}\phi)\psi]$ equals

$$\begin{aligned}&-E_\alpha[(g(\eta_{j_0}) - \eta_{2j_0}g(\eta_0)/\eta_0)\psi]p(j_0) + E_\alpha[(g(\eta_{2j_0}) - \eta_{j_0}g(\eta_0)/\eta_0)\psi]p(-j_0)\\ &\quad+\sum_{j\neq\pm j_0}\Big\{E_\alpha[(g(\eta_{j_0-j}) - \eta_{j_0}g(\eta_0)/\eta_0)\psi] - E_\alpha[(g(\eta_{j_0}) - \eta_{j_0+j}g(\eta_0)/\eta_0)\psi]\Big\}p(j)\\ &\quad-\alpha a_{j_0}E_\alpha[\psi(\eta+\delta_{j_0}) - \psi(\eta+\delta_0)]p(-j_0)\\ &\quad+a_{j_0}E_\alpha[g(\eta_0)(\psi(\eta) - \psi(\tau_{-j_0}(\eta^{0,-j_0})))]p(j_0).\end{aligned}$$

As we can take $E_\alpha[\psi] = 0$ without loss of generality, with Lemma 2.1, $\rho p(j_0)E_\alpha[(\mathcal{L}\phi)\psi]$ equals

$$\begin{aligned}&\rho E_\alpha[(g(\eta_0)/\eta_0)\psi]p(j_0) - E_\alpha[(g(\eta_{j_0}) - \alpha)\psi]p(j_0) + E_\alpha[(\eta_{2j_0} - \rho)(g(\eta_0)/\eta_0)\psi]p(j_0)\\ &\quad+\sum_{j\neq\pm j_0}\Big\{E_\alpha[(g(\eta_{j_0-j}) - \eta_{j_0}g(\eta_0)/\eta_0\psi] - E_\alpha[(g(\eta_{j_0}) - \eta_{j_0+j}g(\eta_0)/\eta_0\psi]\Big\}p(j)\\ &\quad-a_{j_0}E_\alpha[g(\eta_0)\frac{\eta_0-1}{\eta_0}\psi(\eta^{0,j_0}) - \psi(\eta)]p(-j_0)\\ &\quad+a_{j_0}E_\alpha[g(\eta_0)(\psi(\eta) - \psi(\tau_{-j_0}(\eta^{0,-j_0})))]p(j_0)\\ &= \rho p(j_0)E_\alpha[(g(\eta_0)/\eta_0)\psi] + K_1 + K_2 + K_3 + K_4 + K_5.\end{aligned}$$

Hence, to show $\|\mathfrak{h} - \mathcal{L}\phi\|_{-1,\lambda} < \infty$, by the variational characterization (cf. (2.2)), we need only show that

$$|E_\alpha[(\mathfrak{h} - \mathcal{L}\phi)\psi]| \ =\ (\rho p(j_0))^{-1}|K_1 + K_2 + K_3 + K_4 + K_5| \ \le\ C\|\psi\|_1$$

for some constant $C = C(\alpha, p)$. To this end, the terms K_3, K_4 and K_5 are handled analogously as I_2, I_3 and I_4 above in the $d = 1$ case. To bound K_1 and K_2, we invoke the following result.

Proposition 2.2 *Consider $d \geq 3$ reference frame processes such that g satisfies assumption (SP). Let f be a $L^4(Q_\alpha)$ function supported on a finite number of vertices of $\mathbb{Z}^d \setminus \{0\}$ which is mean-zero, $E_\alpha[f] = 0$. Then, $\|f\|_{-1} < \infty$.*

Proof. The proof is virtually the same as for Theorem 1.2 [19]. One can bound in terms of a constant $C = C(f, \alpha, p, d)$ that

$$E_\alpha[f\psi] \;\leq\; C\left[\sum_j \sum_{i\neq 0} E_\alpha[g(\eta_i)(\psi(\eta^{i,i+j}) - \psi(\eta))^2]s(j)\right]^{1/2} \;\leq\; C\|\psi\|_1$$

by straightforwardly avoiding the orign. This relation gives the desired statement. □

Note now Proposition 2.2 directly applies to K_1. For K_2, we first condition on η_0 to get

$$\begin{aligned} |K_2| &= p(j_0)|E_\alpha[E_\alpha[(\eta_{2j_0} - \rho)\psi(\cdot;\eta_0)|\eta_0](g(\eta_0)/\eta_0)]| \\ &\leq p(j_0)E_\alpha[C\|\psi(\cdot;\eta_0)\|_1(g(\eta_0)/\eta_0)] \\ &\leq Cp(j_0)E_\alpha[(g(\eta_0)/\eta_0)^2]^{1/2}\|\psi\|_1 \end{aligned}$$

where $\psi(\cdot;\eta_0)$ denotes ψ as a function of $\{\eta_i : i \neq 0\}$ with η_0 fixed. This finishes the proof of Theorem 2 in this case. □

3 Proof of Theorem 3

We first define the notion of a positively associated stationary increments L^2 process $N(t)$. This is an L^2 process where

$$E[\phi(N(t+s) - N(t))\psi(N(t))] \geq E[\phi(N(s))]E[\psi(N(t))]$$

for all ϕ and ψ increasing. For such processes we have the Newman-Wright result (cf. [11]).

Theorem 4 *Suppose $N(t)$ is an L^2 process with positively associated stationary increments such that the limit exists*

$$lim_{t\to\infty}\frac{1}{t}E\bigg[(N(t) - E[N(t)])^2\bigg] \;=\; \sigma^2 < \infty.$$

Then, we have weak convergence to Brownian motion in Skorohod space,

$$\frac{1}{\sqrt{\lambda}}\bigg(N(t) - E[N(t)]\bigg) \;\to\; \sigma B(t).$$

The strategy will now be to verify that the tagged position $x(t)$ in $d = 1$ under the assumptions of Theorem 3 has associated increments and that its variance scales diffusively so that the Newman-Wright statement applies.

The following is a useful coupling which essentially says adding more particles to the system only slows down the tagged particle.

Lemma 3.1 *Under the assumptions on g and p in dimension $d = 1$ in Theorem 3, we can couple two two copies of the joint process, $(x^1(t), \xi^1(t))$ and $(x^2(t), \xi^2(t))$ where $\xi^1(0) \le \xi^2(0)$ coordinatewise and $x^1(0) \ge x^2(0)$ so that at all later time t, $x^1(t) \ge x^2(t)$.*

Proof. We make the coupling so that when an ξ^1 particle moves, a corresponding ξ^2 particle also moves to the right, and also when x^2 would move ahead of x^1 then x^1 also moves.

More carefully, at vertex $x \neq x^1, x^2$, the basic coupling applies–with rate $g(\xi^1_x)$ a particle from x in both systems moves; and with rate $g(\xi^2_x) - g(\xi^1_x)$ a particle from x in system 2 moves.

When $x^1 \neq x^2$, with rate $g(\xi^1_{x^1})(\xi^1_{x^1} - 1)/\xi^1_{x^1}$ a non-tagged particle in system 1 and a particle in system 2 moves from location x^1; with rate $g(\xi^1_{x^1})/\xi^1_{x^1}$ the tagged particle from system 1 and a particle from system 2 at x^1 moves; and with rate $g(\xi^2_{x^1}) - g(\xi^1_{x^1})$ a particle in system 2 at x^1 moves.

With respect to location x^2, with rate $g(\xi^1_{x^2})(\xi^1_{x^2} - 1)/\xi^1_{x^2}$ a particle from system 1 and a non-tagged particle in system 2 moves from x^2; with rate $g(\xi^2_{x^2})/\xi^2_{x^2}$ the tagged particle in system 2 and a particle in system 1 move from location x^2; with rate $g(\xi^1_{x^2})/\xi^1_{x^2} - g(\xi^2_{x^2})/\xi^2_{x^2}$ a particle in system 1 moves from x^2; with rate $g(\xi^2_{x^2})(\xi^2_{x^2} - 1)/\xi^2_{x^2} - g(\xi^1_{x^2})(\xi^1_{x^2} - 1)/\xi^1_{x^2}$ a non-tagged particle moves from system 2 at x^2.

When $x^1 = x^2 = x$, with rate $g(\xi^1_x)(\xi^1_x - 1)/\xi^1_x$ a non-tagged particle from x in both systems moves; with rate $g(\xi^2_x)/\xi^2_x$ both tagged particles move; with rate $g(\xi^1_x)/\xi^1_x - g(\xi^2_x)/\xi^2_x$ the tagged particle in system 1 and a non-tagged particle in system 2 moves; with (the remaining) rate

$$g(\xi^2_x)\frac{\xi^2_x - 1}{\xi^2_x} - g(\xi^1_x)\frac{\xi^1_x - 1}{\xi^1_x} - \frac{g(\xi^1_x)}{\xi^1_x} + \frac{g(\xi^2_x)}{\xi^2_x} \;=\; g(\xi^2_x) - g(\xi^1_x)$$

a non-tagged particle in system 2 moves.

We omit the generator formulation. □

The next lemma owes some intuition to Theorem 2 [5].

Lemma 3.2 *In $d = 1$, under the assumptions of Theorem 2, the L^2 process $x(t)$ under equilibrium Q_α has positively associated stationary increments.*

Proof. From (1.3), clearly $x(t)$ is an L^2 process. Also under equilibrium Q_α, $x(t)$ has stationary increments. Consider now the sequence, for increasing ϕ and ψ,

$$E_\alpha[\phi(x(t+s) - x(t))\psi(x(t))] \;=\; E_\alpha[\psi(x(t))E_{\eta(t)}[\phi(x(s))]]$$

$$
\begin{aligned}
&= E_\alpha^*[\psi(x^*(0) - x^*(t))E_{\eta^*(0)}[\phi(x(s))]] \\
&= E_\alpha^*[E_{\eta^*(0)}^*[\psi(x^*(0) - x^*(t))]E_{\eta^*(0)}[\phi(x(s))]] \\
&= \int E_{\eta^*}^*[\psi(x^*(0) - x^*(t))]E_{\eta^*}[\phi(x(s))]dQ_\alpha(\eta^*)
\end{aligned}
$$

where in the second step we note $x(0) = 0$, and reverse time at t with $x^*(u) = x(t-u)$, $\eta^*(u) = \eta(t-u)$, and E_α^* and $E_{\eta^*}^*$ denotes expectation with respect to the reversed process with initial distribution Q_α and state η^* respectively.

Consider the functions $E_{\eta^*}^*[\psi(x^*(0) - x^*(t))]$, and $E_{\eta^*}[\phi(x(s))]$ as functions of η^*. Both are decreasing coordinatewise by the coupling in Lemma 3.1. Indeed, from the coupling, we see that, by increasing η^* by one particle, $x^*(0) - x^*(t)$ decreases (recall that the reversed $*$ process moves to the left), and $x(t)$ decreases. In other words, both functions decrease coordinatewise in η^*.

With this monotonicity, the associated property follows from the standard FKG inequality for product measures (see Liggett [9]). □

We now turn to an analysis of the variance. Recall the definition of $\mathfrak{f}$ (cf. near (1.1)).

Lemma 3.3 *In all $d \geq 1$, the variance $V(t) = E_\alpha[|x(t) - E_\alpha[x(t)]|^2]$ satisfies*

$$
V(t) = \frac{\alpha}{\rho}\sum |j|^2 p(j)t + 2\int_0^t E_\alpha[x(s)\cdot \mathfrak{f}(\eta(s))]ds.
$$

Proof. We continue the sequence (1.2). Write

$$
\begin{aligned}
V(t) &= \frac{\alpha}{\rho}\sum |j|^2 p(j)t + 2\int_0^t E_\alpha[M(s)\cdot \mathfrak{f}(\eta(s))]ds + E_\alpha[|A(t)|^2] \\
&= \frac{\alpha}{\rho}\sum |j|^2 p(j)t + 2\int_0^t E_\alpha[x(s)\cdot \mathfrak{f}(\eta(s))]ds - 2E_\alpha[A(s)\cdot \mathfrak{f}(\eta(s))] + E_\alpha[|A(t)|^2] \\
&= \frac{\alpha}{\rho}\sum |j|^2 p(j)t + 2\int_0^t E_\alpha[x(s)\cdot \mathfrak{f}(\eta(s))]ds.
\end{aligned}
$$

Here, in the second line, we use $x(t) - E_\alpha[x(t)] = M(t) + A(t)$, and in the last line that $|A(t)|^2 = 2\int_0^t \int_0^s \mathfrak{f}(\eta(r))\cdot \mathfrak{f}(\eta(s))drds = 2\int_0^t A(s)\cdot \mathfrak{f}(\eta(s))ds$. □

Lemma 3.4 *In $d = 1$, the variance $V(t) = E_\alpha[(x(t) - E_\alpha[x(t)])^2]$ is super-additive, and so the limit $\lim_{t\to\infty} V(t)/t$ exists.*

Proof. We study the term $E_\alpha[x(s)\mathfrak{f}(\eta(s))]$ appearing in the variance expression in Lemma 3.3. Reverse time at s (using the notation given in proof of Lemma 3.2), to obtain

$$
E_\alpha[x(s)\mathfrak{f}(\eta(s))] = E_\alpha^*[(x^*(0) - x^*(s))\mathfrak{f}(\eta^*(0))] \tag{3.1}
$$

Now, write, using (3.1) and the variance decomposition in Lemma 3.3, that

$$\begin{aligned}
V(t+r)-V(r)-V(t) &= 2\int_0^t E^*_\alpha[(x^*(s)-x^*(s+r))\mathfrak{f}(\eta^*(0))]ds \\
&= 2\int_0^t E^*_\alpha[E^*_{\eta^*(s)}[x^*(0)-x^*(r)]\mathfrak{f}(\eta^*(0))]ds \\
&= 2\int_0^t E_\alpha[E^*_{\eta(0)}[x^*(0)-x^*(r)]\mathfrak{f}(\eta(s))]ds \\
&= 2E_\alpha[E^*_{\eta(0)}[x^*(0)-x^*(r)]E_{\eta(0)}[A(t)]]] \\
&= 2E_\alpha[E^*_{\eta(0)}[x^*(0)-x^*(r)]E_{\eta(0)}[x(t)-E_\alpha[x(t)]]].
\end{aligned}$$

Here, we shifted variables s to $s+r$ in the first line, conditioned up to time s in the second line, reversed time at s in the third, and used the martingale decomposition $x(t)-E_\alpha[x(t)] = M(t)+A(t)$ in the last line.

The last product, as in proof of Lemma 3.2, is the product of decreasing functions of η. Now use FKG inequality to finish the proof. $\square$

We are now ready to prove Theorem 3.

Proof of Theorem 3. Under the assumptions of Theorem 3, we can invoke the Newman-Wright principle: By Lemma 3.2, $x(t)$ has positively associated increments. By Lemma 3.4, the limit $\lim_{t\to\infty} V(t)/t = \sup_{t\geq 1} V(t)/t$ exists; and, by Theorem 2, $\sup_{t\geq 1} V(t)/t < \infty$.

Finally, to show the limit $\lim_{t\to\infty} V(t)/t > \alpha/\rho$ we need only show by superadditivity, noting (3.1), that

$$\int_0^1 E_\alpha[x(s)\mathfrak{f}(\eta(0))]ds \ > \ 0. \tag{3.2}$$

But, one can write

$$E_\alpha[x(s)\mathfrak{f}(\eta(s))] \ = \ E_\alpha\left[(x(t)-E_\alpha[x(t)])\frac{g(\eta_0)}{\eta_0}\right] \ = \ \frac{\alpha}{\rho}\left\{E'_\alpha[x(t)]-E_\alpha[x(t)]\right\}$$

after some algebra where E'_α is the process expectation with respect to initial distribution $Q'_\alpha = \prod_{i\neq 0}\mu_\alpha \times \mu'_\alpha$ and

$$\mu'_\alpha(k) \ = \ \frac{1}{Z'_\alpha}\frac{\alpha^{k-1}}{g(1)\cdots g(k-1)} \quad \text{for } k\geq 1$$

with normalization Z'_α. The interpretation is that μ'_α puts a particle at the origin and distributes other particles there according to μ_α. It is a straightforward computation, under Assumption (ID), that $\mu'_\alpha << \mu^0_\alpha$ in stochastic order, and so can couple two joint systems starting from Q_α and Q'_α so that the tagged particle under Q_α always is ahead of its counterpart under Q'_α.

Since the inequality $\mu'_\alpha << \mu_\alpha$ is strict, with positive probability, the Q'_α system has strictly less particles than the Q_α system initially at the origin. It is not hard now to construct a situation with positive probability, as all clocks are exponential, where the tagged particle positions differ at some time $0 < t \leq 1/2$, and that this difference is maintained up to time $t = 1$. Hence, (3.2) holds. $\square$

Last, we discuss briefly here a "particle-level" approach for why $\sup_{t\geq 1} V(t)/t < \infty$ in the case of Theorem 3 should be true. One needs only bound the "drift" part $2\int_0^t E^*_\alpha[(x^*(0) - x^*(s))\mathfrak{f}(\eta^*(s))]ds$, or show

$$E_\alpha[(x(t) - E_\alpha[x(t)])\frac{g(\eta_0(0))}{\eta_0(0)}] \ = \ \frac{\alpha}{\rho}\{E'_\alpha[x(t)] - E_\alpha[x(t)]\} \ = \ O(1).$$

To bound uniformly the difference between the tagged particles in expected value, the key point would be to handle the influence of "extra" particles at the origin in the Q_α system which could "slow down" the Q_α tagged particle and make a large difference. However, these extras, though not quite "second-class" particles, should in the long term behave like them and move much slower than a tagged particle, and so their influence should be negligible in the limit. Making this precise technically however seems difficult. We remark though that these ideas led in part to other interesting "point-of-view shifts" problems [4].

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